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Non-lacunary Gibbs Measures for Certain Fractal Repellers

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Abstract In this paper, we study non-uniformly expanding repellers constructed as the limit sets for a non-uniformly expanding dynamical systems. We prove that given a Hölder continuous potential ϕ satisfying a summability condition, there exists non-lacunary Gibbs measure for ϕ, with positive Lyapunov exponents and infinitely many hyperbolic times almost everywhere. Moreover, this non-lacunary Gibbs measure is an equilibrium measure for ϕ.

Keywords Gibbs measures · Equilibrium states · Thermodynamical formalism · Non-uniform expansion

1 Introduction

The study of relevant invariant measures has produced important developments in Dynamical Systems along the second part of the 20th Century. In particular, arising from a beautiful interplay between Physics and Mathematics, the Thermodynamical Formalism developed in the late 60s and along 70s by Ruelle, Bowen, Sinai, Walters and Parry, among several others important contributors, is one remarkable advance in the study of the Ergodic Theory of hyperbolic systems. Using a rich dictionary between expanding maps or axiom A systems and the statistical mechanics of one dimensional lattices via Markov Partitions, they introduced several important notions in Ergodic Theory brought from Physics, as the notion of *Gibbs measures* and *Pressure* for a dynamical system. Roughly, given an dynamical system f and a potential function ϕ, if we denote by $\mathcal{B}_\epsilon(n, x)$ the set of configurations y such $f^i(y)$ is in

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a neighborhood of size ϵ of $f^i(x)$ for values of $i = 1, 2, \ldots, n$, and by $S_n\phi(x)$ the sum of ϕ along the orbit of x of length n, the Gibbs measure μ_ϕ has an asymptotic distribution of mass given by

$$\mu_\phi\big(\mathcal{B}_\epsilon(n, x)\big) \sim e^{S_n\phi(x) - nP(\phi)} \tag{1}$$

where the symbol $\sim$ means here that the quotient between these quantities differs by a constant that does not depend on x or n, see (4) for details. This measure maximize the *free energy* of the system, i.e., it is a maximum among all invariant probabilities for the expression

$$h_\mu(f) + \int \phi\, d\mu.$$

The maximum of the expression above coincides with the *pressure* $P(\phi)$ of ϕ, by the variational principle. Summarizing, the Gibbs measure is the unique *equilibrium state* for ϕ.

As a consequence of this theory, several important results were obtained for expanding maps and hyperbolic diffeomorphisms. For example, for an expanding map, the asymptotic distribution of periodic points is given by the Gibbs measure associated to the zero potential. Another remarkable result is the existence and uniqueness of the physical measure, that coincides with a Gibbs measure associated to the unstable Jacobian of f. If we assume that the dynamical system has a limit set that is a uniform expanding conformal repeller, it is possible to obtain dimensional estimates for this repeller using a special Gibbs distribution associated to a zero for the called *Bowen's Equation* for f.

Trying to extend this theory beyond the uniform hyperbolic setting, several interesting results were obtained by many authors in a wide variety of situations. Bruin, Keller [2] and Denker, Urbański [6], for special classes of transformations, such as interval maps, rational functions of the sphere, and Hénon-like maps; Buzzi, Sarig [3, 4], Sarig [15], and Yuri [18], for countable Markov shifts and piecewise expanding maps; and Leplaideur, Rios [10] for "horseshoes with tangencies" at the boundary of hyperbolic systems, to mention just a few of the most recent works. In [13], Oliveira and Viana proved existence and uniqueness of equilibrium states using the notion of *non-lacunary* Gibbs measure, introduced there. Roughly, a non-lacunary Gibbs measure is a probability that satisfies (1) for a sequence $n_i = n_i(x)$ of values of n, such that $n_{i+1}/n_i \to 1$. The existence of such measures was studied in [16] and [12]. However, the global picture is still very much incomplete.

In this paper we contribute a sufficient condition for existence of non-lacunary Gibbs measure, applicable to a large class of non-uniformly expanding maps. Under weaker assumptions on the potential than [13] and [16] (see condition (A_2) in Sect. 2), we are able to prove that the non-lacunary Gibbs measure is an equilibrium measure for ϕ. This provides a broader class of potentials that admits equilibrium states and non-lacunary Gibbs measures than [13] and [16]. Another important advance here is that we are constructing non-lacunary Gibbs measures in (possibly) fractal sets, extending [13] in this aspect also.

Some examples that we have in mind are the non-uniformly expanding local diffeomorphisms of Alves, Bonatti, Viana [1] and Hopf-like bifurcations, studied in [8]. These bifurcations occur in several phenomena in applied science as in the Hodgkin-Huxley model for nerve membrane, the Selkov model of glycolysis, the Belousov-Zhabotinsky reaction, the Lorenz attractor and in the Brusselator, a simple chemical system. We are able to show in [7] that in a Hopf-like bifurcation $(f_\mu)_\mu$, for parameters near the bifurcation, it is possible to find an open set of values of γ in the family $\phi_\gamma(x) = -\gamma \log|\det Df_\mu(x)|$ in such a way that ϕ_γ *does not* satisfy conditions in [13] or [16], but satisfy our hypothesis. We use this to get dimensional estimates in the repeller arising from $(f_\mu)_\mu$.

2 Statement of Results

2.1 Non-uniformly Expanding Maps with Holes

Here we describe the abstract model that we consider along this paper. Let $f : M \to M$ be a map on a d-dimensional Riemannian manifold, $d \geq 1$ such that

(A_1) There exist a family of compact path-connected sets $\mathcal{R} = \{R_1, \dots, R_m\}$ in M, with disjoint two-by-two interiors and with finite inner diameter, such that the restriction of f to a neighborhood of each R_i is a C^1 diffeomorphism onto some domain W_i. We assume that for every $j = 1, 2, \dots, m$, if $\operatorname{int} R_j \cap W_i \neq \emptyset$ then $R_j \subset W_i$.

Define Λ as the *repeller* of f in $R_1 \cup \cdots \cup R_m$, i.e., Λ is the set of points whose forward orbits never leaves $R_1 \cup \cdots \cup R_m$:

$$\Lambda = \{x \in M : f^n(x) \in R_1 \cup \cdots \cup R_m \text{ for every } n \geq 0\}.$$

Given $n \geq 1$, we call *n-cylinder* any set of the form

$$R(i_0, \dots, i_{n-1}) = R_{i_0} \cap f^{-1}(R_{i_1}) \cap \cdots \cap f^{-n+1}(R_{i_{n-1}})$$

with $i_0, i_1, \dots, i_{n-1}$ in $\{1, \dots, m\}$. The family of all n-cylinders is denoted by $\mathcal{R}^n$. It is easy to see that for each $n \geq 1$ the set of all n-cylinders form a covering of the repeller Λ.

For each $n \geq 1$ and $i_0, i_1, \dots, i_{n-1}$ in $\{1, \dots, m\}$, we consider the *average least expansion* for the cylinder $R(i_0, i_1, \dots, i_{n-1})$:

$$\psi_n(i_0, i_1, \dots, i_{n-1}) = \frac{1}{n} \sum_{j=1}^{n} \inf_{x \in C_j} \log \left\| Df^{-1}\left(f^j(x)\right) \right\|^{-1},$$

where the infimum is taken over all x in $C_j = R(i_0, \dots, i_{j-1})$. Throughout, $Df^{-i}(f^j(y))$ is to be understood as the inverse of the derivative $Df^i(f^{j-i}(y))$, for any y and $j \geq i$. Note that if

$$\psi_n(i_0, i_1, \dots, i_{n-1}) > c > 0 \tag{2}$$

implies that the derivative Df^n expands every vector:

$$\|Df^{-n}(f^n(x))\| \leq \prod_{j=1}^{n} \|Df^{-1}(f^j(x))\| \leq e^{-cn} \qquad \text{for all } x \in R(i_0, \dots, i_{n-1}).$$

Denote by ω the *lowest contraction rate* in Λ:

$$\omega = \sup_{x \in \Lambda} \|Df(x)^{-1}\|$$

If $\omega < 1$, Λ is a uniformly expanding repeller. In some applications, including Hopf bifurcations discussed at [8], we may have ω greater than 1 and close to 1. We assume here that $\omega > 1$, otherwise Λ is an expanding repeller.

Given a function $\phi : M \to \mathbb{R}$, we write $S_n\phi(x) = \sum_{j=0}^{n-1} \phi(f^j(x))$ and if A is a subset of M, we write $S_n\phi(A) = \sup_{x \in A} S_n\phi(x)$. We assume that ϕ is Hölder continuous and

(A_2) there exist $c > 0$ and $c_1 < P(\phi) - d \log \omega$ such that if $\mathcal{Q}_n(c)$ denote the set of cylinders such that $\psi_n(i_0, i_1, \dots, i_{n-1}) \le c$, for every large n, we have:

$$\sum_{C \in \mathcal{Q}_n(c)} e^{S_n \phi(C)} \le e^{c_1 n}. \tag{3}$$

We fix $c > 0$ as in Hypothesis (A_2) above. Let H be the subset of Λ of points of cylinders that satisfy (2) for infinitely many n. That is

$$H = \bigcap_{k=1} \bigcup_{n \ge k} \left(\bigcup_{\psi_n(i_0, \dots, i_{n-1}) > c} R(i_0, \dots, i_{n-1}) \right).$$

We recall that a measure μ is said a *Gibbs measure* for ϕ and a constant P, if there exists $K > 0$ such that for μ_ϕ almost every point $x \in \Lambda$

$$K^{-1} \le \frac{\mu_\phi(R^n(x))}{e^{S_n \phi(x) - n P(\phi)}} \le K. \tag{4}$$

In [13], the authors introduced the notion of *non-lacunary* Gibbs measure. They defined a non-lacunary Gibbs measure (for ϕ and P) as any measure μ_ϕ such that for almost every point $x \in \Lambda$, (4) above holds for a sequence $n_i = n_i(x)$ such that n_{i+1}/n_i converges to one.

Under more restrictive conditions on ϕ than ours, in [13] the authors were able to prove that non-lacunary Gibbs measures exists and they are *equilibrium states* for f and ϕ. If μ_ϕ denote a non-lacunary Gibbs measure, then

$$h_{\mu_\phi}(f) + \int \phi \, d\mu_\phi = P(\phi).$$

Here, $P(\phi)$ is the *pressure* of ϕ and it is defined by

$$P(\phi) = \sup \left\{ h_\mu(f) + \int \phi \, d\mu \right\}, \tag{5}$$

where the supremum is taken over all invariant probabilities. Now, we state the main result in this paper:

Theorem A *Let $f : M \to M$ be a C^1 map satisfying (A_1) and ϕ a Hölder continuous potential satisfying (A_2). If f is transitive on Λ, there exists a non-lacunary Gibbs measure μ_ϕ for ϕ. Moreover, the support of μ_ϕ coincides with $\overline{H}$ and μ_ϕ is an equilibrium state of f.*

Remark 2.1 In a recent preprint (see [11]), the author was able to prove that if $P(\phi) > P_{\Lambda \setminus H}(\phi)$ (see Sect. 3.2) and under the hypothesis that exists some conformal expanding measure for ϕ (see Sect. 3.3), there exists a unique equilibrium measure for ϕ. In particular, follows from our method and proofs, joint with this result, that the measure μ_ϕ above is the unique equilibrium measure for ϕ.

3 Some Useful Tools

We devote this section to develop some tools that we use in the proof of Theorem A. The first one is the notion of uniform expansion along orbits:

3.1 Hyperbolic Times

Definition 3.1 We say that $n \in \mathbb{N}$ is a *c-hyperbolic time* for $x \in M$ if there exists $d > c$ such that

$$\prod_{k=0}^{j-1} \|Df(f^{n-k}(x))^{-1}\| \le e^{-dj} \quad \text{for every } 1 \le j \le n.$$

Here, c is fixed and we just say *hyperbolic time* instead of c-hyperbolic time.

Given a partition of M, let $\mathcal{R}^n$ be the partition of M into length-n cylinders. For each point $x \in M$ we denote by $R^n(x) \in \mathcal{R}^n$ some atom that contains x.

Definition 3.2 We say that n is a *hyperbolic time for a cylinder* $R^n \in \mathcal{R}^n$ if n is a hyperbolic time for every $x \in R^n$ and in this case we say that R^n is a *hyperbolic cylinder*. We denote by $\mathcal{R}_h^n$ the set of the cylinders $R^n \in \mathcal{R}^n$ for which n is a hyperbolic time and by $\mathcal{R}_h = \bigcup_{n\ge 1} \mathcal{R}_h^n$.

Next lemma is a consequence of Pliss' Lemma:

Lemma 3.3 *If $R(i_0, \dots, i_{n-1})$ satisfy $\psi_n(i_0, \dots, i_{n-1}) > c$, there exist a real number $\theta > 0$ and integers numbers $1 \le n_1 < \dots < n_k \le n$ such that $k \ge \theta n$ and $R(i_0, \dots, i_{n_k-1})$ is a c-hyperbolic cylinder.*

Proof This is a direct consequence of Pliss' Lemma. See [1] for a proof. □

The main property of a hyperbolic cylinder R^n is that inverse branches of f^n contracts distances uniformly on R^n. More precisely:

Lemma 3.4 *If $R^n \in \mathcal{R}_h^n$ then for every $x, y \in R^n$ and $1 \le j \le n$*

1. $d(f^{n-j}(x), f^{n-j}(y)) \le e^{-jc} d(f^n(x), f^n(y))$;
2. *If ϕ is (C, α)-Hölder continuous, there exists a constant K_1 that depends only on f and C such that*:

$$|S_n\phi(x) - S_n\phi(y)| \le K_1 d(f^n(x), f^n(y))^\alpha.$$

Proof See [13], Lemma 3.5. □

As a consequence of the result above, we obtain control of the Jacobian of f at a hyperbolic time:

Corollary 3.5 *There exists $K_2 > 0$ such if $R^n \in \mathcal{R}_h^n$ then for all $x, y \in R^n$*

$$K_2^{-1} \le \frac{e^{S_n\phi(x)}}{e^{S_n\phi(y)}} \le K_2.$$

Proof By Lemma 3.4 above, we have that

$$\frac{e^{S_n\phi(x)}}{e^{S_n\phi(y)}} = e^{S_n\phi(y) - S_n\phi(x)} \le e^{K_1 d(f^n(x), f^n(y))^\alpha}.$$

Then we just need to choose K_2 bigger than the $\sup\{\mathrm{diam}(R_i)\}$. The opposite inequality holds in a similar fashion. □

3.2 Pressure

In this section we study the relation between the pressure of a potential and the set H. We will see that if ϕ satisfy (A_2), the supremum defined at (5) can be considered as a supremum over measures supported on H.

One extra complication here is the fact that we need to use the notion of pressure *relative* to H. To define $P_H(\phi)$, the pressure of ϕ relative to H, we need to consider a more general definition of pressure that include that non-compact case, since H is a non-compact invariant set.

To define this notion, let X be any subset of Λ that is invariant under f, and let $\mathcal{U}$ be a cover of Λ. To each finite sequence $(U_0, \dots, U_{n-1})$ of elements of $\mathcal{U}$, associate the set

$$U = \{x \in \Lambda : x \in U_0, f(x) \in U_1, \dots, f^{n-1}(x) \in U_{n-1}\} \tag{6}$$

and write $n(U) = n$. Given any $N \geq 1$, define $\mathcal{S}_N(\mathcal{U})$ to be the family of all sets U of this form, for all values of $n(U) \geq N$.

Given any $\alpha \in \mathbb{R}$, consider the number

$$m_X(\phi, \alpha, \mathcal{U}, N) = \inf_{\mathcal{G}} \sum_{U \in \mathcal{G}} \exp\left(S_{n(U)}\phi(U) - \alpha n(U)\right) \tag{7}$$

where the infimum is taken over all families $\mathcal{G} \subset \mathcal{S}_N(\mathcal{U})$ that cover X. We have a monotone non-decreasing sequence in N. Define

$$m_X(\phi, \alpha, \mathcal{U}) = \lim_{N \to \infty} m_X(\phi, \alpha, \mathcal{U}, N).$$

It is not difficult to see that there exists a unique real number $P_X(\phi, \mathcal{U})$ satisfying

$$\begin{aligned} P_X(\phi, \mathcal{U}) &= \inf\{\alpha : m_X(\phi, \alpha, \mathcal{U}) = 0\} \\ &= \sup\{\alpha : m_X(\phi, \alpha, \mathcal{U}) = +\infty\}. \end{aligned}$$

Definition 3.6 The *pressure of f for ϕ relative to X* is

$$P_X(\phi) = \lim_{\operatorname{diam}\mathcal{U} \to 0} P_X(\phi, \mathcal{U}).$$

Theorem 11.1 in [14] states that the limit does exist, that is, given any sequence of covers $\mathcal{U}_k$ of X with diameter going to zero, $P_X(f, \phi, \mathcal{U}_k)$ converges and the limit does not depend on the choice of the sequence.

Let $\mathcal{I}_X$ denote the set of invariant probability measures η such that $\eta(X) = 1$. If X is a compact set then (see [17, Theorem 9.10] or [14, Theorem A2.1])

$$P_X(\phi) = \sup\left\{h_\eta(f) + \int \phi \, d\eta : \eta \in \mathcal{I}_X\right\}.$$

This applies, in particular, when $X = \Lambda$. We just write $P(\phi)$ to mean $P_\Lambda(\phi)$. In the general non-compact case one inequality remains true:

$$P_X(\phi) \geq \sup\left\{h_\eta(f) + \int \phi \, d\eta : \eta \in \mathcal{I}_X\right\}. \tag{8}$$

In particular, if $\mathcal{I}_X$ contains some equilibrium state then the equality holds in (8), and $P_X(\phi) = P(\phi)$.

In the next lemma, we prove that it is possible to estimate the pressure of H as the limits of the pressure of the set H with respect to the cover by hyperbolic cylinders $\mathcal{U}$. Here, we follow [14] (see Theorem 11.1).

Lemma 3.7 *If $\mathcal{U}^k$ is the cover of H by cylinders $U \in \mathcal{R}_h^i$ for $i \geq k$, then the pressure of H is given by*

$$P_H(\phi) = \lim_{k \to +\infty} P_H(\phi, \mathcal{U}^k).$$

Proof Fix a finite open cover $\mathcal{W}$ of Λ. If $\mathcal{V}$ is a cover of H with diameter less or equal to the Lebesgue number of $\mathcal{W}$, then for every element $V \in \mathcal{V}$ there exists an element $W = W(V)$ such $V \subset W$.

In particular, every cylinder $V^n = V_1 \cap f^{-1}(V_2) \cap \ldots \cap f^{-n+1}(V_n)$ of length n of $\mathcal{V}_n$ is contained in some element

$$W^n(V^n) = W(V_1) \cap f^{-1}(W(V_2)) \cap \ldots \cap f^{-n+1}(W(V_n))$$

of $\mathcal{W}_n$. Thus, given $\mathcal{G}$ a cover of H by elements of $\bigcup_{n \geq N} \mathcal{V}_n$, we may consider the corresponding cover $\mathcal{F}(\mathcal{G})$ of elements in $\bigcup_{n \geq N} \mathcal{W}_n$. Given $W \in \mathcal{W}$, define

$$\kappa_W = \sup\{|\phi(x) - \phi(y)|; x, y \in W\}$$

and

$$\kappa_{\mathcal{W}} = \sup_{W \in \mathcal{W}} \kappa_W.$$

Observe that if $\mathcal{G}$ is a cover of H by elements of $\bigcup_{n \geq N} \mathcal{V}_n$ then:

$$\sum_{W \in \mathcal{F}(\mathcal{G})} \exp(S_{n(W)}\phi(W) - \alpha n(W)) \leq \sum_{V \in \mathcal{G}} \exp(S_{n(V)}\phi(V) - (\alpha - \kappa(\mathcal{V}))n(V)).$$

Taking the infimum over the covers $\mathcal{G}$ of H in $\mathcal{V}$ such that $n(U) \geq N$ for all $V \in \mathcal{G}$, we have that:

$$m_H(\phi, \alpha, \mathcal{W}, N) \leq m_H(\phi, \alpha - \kappa(\mathcal{W}), \mathcal{V}, N),$$

and consequently

$$m_H(\phi, \alpha, \mathcal{W}) \leq m_H(\phi, \alpha - \kappa(\mathcal{W}), \mathcal{V}).$$

Thus, given α_0, if $m_H(\phi, \alpha_0, \mathcal{W}) = +\infty$ then $m_H(\phi, \alpha_0 - \kappa(\mathcal{W}), \mathcal{V}) = +\infty$. Hence

$$P_H(\phi, \mathcal{V}) \geq \alpha_0 - \kappa(\mathcal{W}).$$

Taking the supremum of all α_0 such that $m_H(\phi, \alpha_0, \mathcal{W}) = +\infty$, we have

$$P_H(\phi, \mathcal{V}) \geq P_H(\phi, \mathcal{W}) - \kappa(\mathcal{W}). \tag{9}$$

Now, we are in condition to prove the statement in lemma. First, observe that the continuity of ϕ implies that $\kappa(\mathcal{W}) \to 0$ when $\text{diam}(\mathcal{W}) \to 0$.

Fix a sequence $\mathcal{W}^n$ such that $P_H(\phi, \mathcal{W}^n) \to P_H(\phi)$ and $\operatorname{diam} \mathcal{W}^n \to 0$. Since $\mathcal{U}^k$ is a cover by hyperbolic cylinders, we have by Lemma 3.4 that the diameter of the cover $\mathcal{U}^k$ is bounded by e^{-ck}, for some constant K. In particular, we have that $\operatorname{diam}(\mathcal{U}^k) \to 0$ when $k \to \infty$. Then, we may choose $k(n)$ such the diameter of $\mathcal{U}^{k(n)}$ is less than the Lebesgue number of $\mathcal{W}^n$. Thus, by (9) we have that:

$$P_H(\phi, \mathcal{U}^{k(n)}) \geq P_H(\phi, \mathcal{W}_n) - \kappa(\mathcal{W}^n).$$

Since $\operatorname{diam} \mathcal{W}^n \to 0$ when n goes to infinity, we have that $\lim_{n\to\infty} \kappa(\mathcal{W}^n) = 0$. Then:

$$P_H(\phi) = \lim_{n\to\infty} P_H(\phi, \mathcal{W}^n) \leq \lim_{n\to\infty} P_H(\phi, \mathcal{U}^{k(n)}) + \kappa(\mathcal{W}^n) = P_H(\phi).$$

This finishes the proof. □

Now, we prove that under assumption (A_2) the complement of H in Λ can not carry any equilibrium measure. To prove this we need two auxiliaries lemmas. The first one is:

Lemma 3.8 *Let M be a compact manifold of dimension d. There exists a sequence $(\mathcal{T}_k)_k$ of finite triangulations of M and there exist positive constants K_3 and K_4 such that* $\operatorname{diam}(\mathcal{T}_k) \leq K_3 2^{-k}$ *and, given $A \geq 1$, any set $E \subset M$ such that* $\operatorname{diam}(E) \leq A \operatorname{diam}(\mathcal{T}_k)$ *intersects at most $K_4 A^d$ atoms of $\mathcal{T}_k$.*

Proof See [13, Lemma 6.5]. □

Let $(\mathcal{T}_k)_{k\geq 1}$ be as in the previous lemma and, for each $j \geq 1$, denote

$$\mathcal{T}_k^{\ell,j} = \left\{T_0 \cap f^{-\ell}(T_1) \cap \ldots \cap f^{-\ell(j-1)}(T_{j-1}) : T_i \in \mathcal{T}_k \text{ for } 0 \leq i < j\right\}.$$

The crucial estimate in the proof that $P_H(\phi) > P_{\Lambda\setminus H}(\phi)$ (see Corollary 3.10 below) is given in the next lemma.

Lemma 3.9 *Given ℓ, j, and k natural numbers, there exists a family $\mathcal{G}_{\ell,j,k} \subset \mathcal{T}_k^{\ell,j}$ such that:*

1. *for every L, the union $\cup_{j\geq L} \mathcal{G}_{\ell,j,k}$ covers the set $\Lambda \setminus H$;*
2. *there exist $k_0(\ell)$, constant $P < P(\phi)$, and a sequence γ_k that converges to zero, such that for all L big enough, if $k \geq k_0(\ell)$*

$$\sum_{j\geq L} \sum_{G\in\mathcal{G}_{\ell,j,k}} e^{S_{\ell j}\phi(G)} < e^{(Pl+\log K_4+\gamma_k l)L} \tag{10}$$

Proof We define $\mathcal{G}_{\ell,j,k}$ as the family of all elements of $\mathcal{T}_k^{\ell,j}$ that intersect some element $Q \in \mathcal{Q}_n(c)$ such that $\ell(j-1) \leq n < \ell j$. Since for every ℓ, j and k the set $\mathcal{T}_k^{\ell,j}$ covers M, given any $L \geq 1$, the union of $\mathcal{G}_{\ell,j,k}$ over all $j \geq L$ covers $\Lambda \setminus H$, as stated in Part 1.

Now we claim that, for large k and j, there are at most $K_4^j \omega^{d\ell j} \# \mathcal{T}_k$ elements of $\mathcal{T}_k^{\ell,j}$ that intersect any given $Q = Q(i_0, \ldots, i_{n-1})$ as before. Indeed, let

$$T_0 \cap f^{-\ell}(T_1) \cap \ldots \cap f^{-\ell(j-1)}(T_{j-1}) \in \mathcal{T}_k^{\ell,j}$$

be any such element. Then, $T_s \cap f^{-\ell}(T_{s+1})$ intersects $Q(i_{s\ell}, \ldots, i_{(s+1)\ell-1})$ for every $s = 0, 1, \ldots, j-2$. Condition (A_1) implies that f^ℓ is injective on a neighborhood of any element of $\mathcal{R}^\ell$ and Lemma 3.8 gives that the diameter of $\mathcal{T}_k$ goes to zero when $k \to \infty$. So, taking k larger than some function $k_0(\ell)$, we can ensure that $f^{-\ell}(T_{s+1})$ has exactly one connected component, that we denote C_{s+1}, that intersects the neighborhood of radius $\operatorname{diam}(\mathcal{T}_k)$ around $Q(i_{s\ell}, \dots, i_{(s+1)\ell-1})$. By the definition of ω, we have that $\|Df^{-\ell}\| \le \omega^\ell$, and so

$$\operatorname{diam}(C_{s+1}) \le \omega^\ell \operatorname{diam}(T_{s+1}) \le \omega^\ell \operatorname{diam}(\mathcal{T}_k).$$

Then, by Lemma 3.8, C_{s+1} intersects at most $K_4\omega^{\ell d}$ atoms of $\mathcal{T}_k$. Applying this argument, successively, to $s = j-2, \dots, 1, 0$, we conclude that there are at most $\#\mathcal{T}_k\big(K_4\omega^{\ell d}\big)^{j-1}$ sequences $(T_0, \dots, T_{j-1})$ as we have been considering.

Now, we obtain inequality (10) in Part 2. Define $C = \sup_{x\in\Lambda} \max_{0\le i\le l-1} e^{S_i\phi(x)}$. We have

$$e^{S_{\ell j}\phi(Q)} \le Ce^{S_n\phi(Q)}. \tag{11}$$

Let $(\gamma_k)_k$ be the sequence of real numbers defined by

$$\gamma_k = \log \sup_{d(x,y)<1/k} \frac{e^{\phi(x)}}{e^{\phi(y)}}.$$

Note that $\lim \gamma_k = 0$. Given $G \in \mathcal{G}_{\ell,k,j}$ that intersects $Q \in \mathcal{Q}_n(c)$ with $\ell(j-1) \le n < \ell j$, we have

$$e^{S_{\ell j}\phi(G)} \le e^{\gamma_k \ell j} e^{S_{\ell j}\phi(Q)}.$$

Using the estimate on the number of elements of $\mathcal{G}_{\ell,k,j}$ obtained above, we have

$$\sum_{G\in\mathcal{G}_{\ell,j,k}} e^{S_{\ell j}\phi(G)} \le \sum_{n=\ell(j-1)}^{\ell j-1} \sum_{Q\in\mathcal{Q}_n(c)} \#\mathcal{T}_k K_4^j \omega^{d\ell j} e^{\gamma_k \ell j} e^{S_{\ell j}\phi(Q)}.$$

It follows from (11) and hypothesis (A_2) that

$$\begin{aligned}
\sum_{G\in\mathcal{G}_{\ell,j,k}} e^{S_{\ell j}\phi(G)} &\le \sum_{n=\ell(j-1)}^{\ell j-1} \sum_{Q\in\mathcal{Q}_n(c)} C\#\mathcal{T}_k K_4^j \omega^{d\ell j} e^{\gamma_k \ell j} e^{S_n\phi(Q)} \\
&\le \sum_{n=\ell(j-1)}^{\ell j-1} C\#\mathcal{T}_k K_4^j \omega^{d\ell j} e^{\gamma_k \ell j} e^{c_1 n} \\
&\le \ell C\#\mathcal{T}_k K_4^j \omega^{d\ell j} e^{\gamma_k \ell j} e^{c_1 \ell j}.
\end{aligned}$$

Thus, taking $C(\ell) = \ell C\#\mathcal{T}_k$, we have

$$\sum_{j\ge L} \sum_{G\in\mathcal{G}_{\ell,j,k}} e^{S_{\ell j}\phi(G)} \le \sum_{j\ge L} C(l) e^{(\log K_4 + (d\log\omega + c_1)\ell + \gamma_k \ell)j}.$$

Since by hypothesis (A_2) we have that $c_1 < P(\phi) - d\log\omega$, it is possible to take P such that $d\log\omega + c_1 < P < P(\phi)$ and for L big enough and $k > k_0(\ell)$ the inequality (10) holds. This finish the proof of the lemma. □

Corollary 3.10 $P_{\Lambda\setminus H}(\phi) < P(\phi)$.

Proof Here one uses the estimate obtained in the Part 2 of Lemma 3.9 to calculate the pressure of ϕ with respect to f^ℓ, and the fact that $P_H(S_\ell\phi, f^\ell) = \ell P_H(\phi, f)$. By (7),

$$m_{\Lambda\setminus H}(f^\ell, S_l\phi, \alpha, \mathcal{T}_k, L) \le \sum_{j\ge L}\sum_{G\in\mathcal{G}_{\ell,j,k}} e^{-\alpha j + S_{\ell j}\phi(G)}.$$

For L big enough and $k > k_0(\ell)$, the inequality (10) gives some number $P < P(\phi)$ such that

$$m_{\Lambda\setminus H}(f^\ell, S_l\phi, \alpha, \mathcal{T}_k, L) \le e^{(-\alpha + P\ell + \log K_4 + c_k\ell)L}.$$

This imply that

$$P_{\Lambda\setminus H}(f^\ell, S_\ell\phi, \mathcal{T}_k) \le (P + c_k)\ell + \log K_4.$$

Taking the limit when $k\to\infty$, and recalling that $\operatorname{diam}\mathcal{T}_k$ goes to zero, we have that $P_{\Lambda\setminus H}(f^\ell, S_\ell\phi) \le P\ell + \log K_4$. This gives that,

$$P_{\Lambda\setminus H}(f,\phi) \le P + \frac{1}{\ell}\log K_4.$$

Taking the limit when ℓ goes to infinity, we have that $P_{\Lambda\setminus H}(\phi) \le P < P(\phi)$. □

Corollary 3.11 $P(\phi) = P_H(\phi)$.

Proof By the Theorem 11.2 of [14], $P(\phi) = \sup\{P_H(\phi), P_{\Lambda\setminus H}(\phi)\}$ and by Corollary 3.10 we have that $P_{\Lambda\setminus H}(\phi) < P(\phi)$. □

3.3 Conformal Measures

The *Jacobian* of a measure η with respect to f is the (essentially unique) function $J_\eta f$ satisfying

$$\eta(f(A)) = \int_A J_\eta f\, d\eta,$$

for any measurable set $A\subset\Lambda$ such that $f|_A$ is injective. In other words, the Jacobian is defined by $J_\eta f = d(f_*\eta)/d\eta$. Jacobians need not exist, in general, but if f is at most countable-to-one then $J_\eta f$ does exist for every f-invariant measure.

Definition 3.12 We call *conformal measure* associated to ϕ and λ, any measure ν such that $J_\nu f = \lambda e^{-\phi}$.

Now, we introduce a helpful tool in order to produce a conformal measure. The *transfer operator* $\mathcal{L}_\phi : C^0(\Lambda)\to C^0(\Lambda)$ is defined by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y). \tag{12}$$

Definition 3.13 The *spectral radius* of $\mathcal{L}_\phi$ is the number

$$r(\mathcal{L}_\phi) = \limsup \|\mathcal{L}_\phi^n\|^{\frac{1}{n}}.$$

Observe that $\mathcal{L}_\phi$ is positive and $r(\mathcal{L}_\phi) = \limsup \|\mathcal{L}_\phi^n 1\|^{\frac{1}{n}}$. We prove that:

Lemma 3.14 *There exists some conformal measure associated to ϕ and $r(\mathcal{L}_\phi)$.*

Proof Fix $\lambda = r(\mathcal{L}_\phi)$. Let $K = \{g \in C^0(\Lambda); \min g(x) > 0\}$ be the cone of positive functions and

$$V = \{\lambda\varphi - \mathcal{L}_\phi(\varphi) : \varphi \in C^0(\Lambda)\}.$$

Clearly, V is a linear subspace and K is a open convex set. We claim that V and K are disjoint. Indeed, suppose $\psi = \lambda\varphi - \mathcal{L}_\phi(\varphi)$ belongs to K, for some $\varphi \in C^0(\Lambda)$. There exists $\delta > 0$ such that $\delta \max(-\varphi) \le \min \psi$. Then

$$\mathcal{L}_\phi(-\varphi) = -\lambda\varphi + \psi \ge (\lambda + \delta)(-\varphi).$$

Since $\mathcal{L}_\phi$ is a positive operator, it follows that $\mathcal{L}_\phi^n(-\varphi) \ge (\lambda+\delta)^n(-\varphi)$ for every $n \ge 1$. This implies that the spectral radius of $\mathcal{L}_\phi$ is at least $\lambda + \delta$, contradicting the definition of λ. This contradiction proves that $K \cap V = \emptyset$, as we claimed. Then, by Mazur's Separation Theorem (see [5, Proposition 7.2]), there exists some continuous linear functional $\nu : C^0(\Lambda) \to \mathbb{R}$ such that

$$\int \varphi \, d\nu > 0 \quad \text{for every } \varphi \in K \quad \text{and} \quad \int \varphi \, d\nu = 0 \quad \text{for every } \varphi \in V.$$

The first property means (by Riez-Markov Theorem) that the restriction of ν to continuous functions is a positive measure and so, up to normalization, we may suppose it is a probability.

Now, we show that ν is conformal. Indeed, let A be any measurable set such that $f \mid A$ is injective. Observe that

$$\mathcal{L}_\phi(e^{-\phi}\chi_A)(x) = \sum_{f(y)=x} e^{\phi(y)} e^{-\phi(y)} \chi_A(y) = \sum_{f(y)=x} \chi_A(y).$$

The last expression is equal to $\chi_{f(A)}(x)$, because $f \mid A$ is injective. Hence,

$$\int \lambda e^{-\phi} \chi_A \, d\nu = \int \mathcal{L}_\phi(e^{-\phi}\chi_A) \, d\nu = \nu(f(A)).$$

Thus,

$$\nu(f(A)) = \int_A \lambda e^{-\phi} d\nu,$$

which proves the lemma. □

Remark 3.15 It is straightforward to check that if the Jacobian of ν with respect to f is $J_\nu f = \lambda e^{-\phi}$, then the Jacobian of ν with respect to f^n is $J_\nu f^n = \lambda^n e^{-S_n\phi}$, for every $n \in \mathbb{N}$.

Proposition 3.16 *Let ϕ be a Hölder continuous map and ν conformal measure for ϕ. If $P = \log r(\mathcal{L}_\phi)$, there exists $K_5 > 0$ such that for any $R^n \in \mathcal{R}_h^n$ and $x \in R^n$ then*

$$K_5^{-1} \le \frac{\nu\big(R^n(x)\big)}{\exp\big(-nP + S_n\phi(x)\big)} \le K_5 \tag{13}$$

Proof By Remark 3.15, the Jacobian of f^n is given by $J_\nu f^n = e^{nP-S_n\phi}$. Hence, if $R^n = R(i_0,\dots,i_{n-1})$, by hypothesis (A_1)

$$\nu(f(R_{i_{n-1}})) = \nu(f^n(R^n)) = \int_{R^n} J_\nu f^n d\nu = \int_{R^n} e^{nP-S_n\phi(x)}\, d\nu(x).$$

By Corollary 3.5, there exists K_2 not depending on n such for every $x, y \in R^n$

$$K_2^{-1} J_\nu f^n(y) \le J_\nu f^n(x) \le K_2 J_\nu f^n(y).$$

It follows that

$$K_2^{-1}\nu(f(R_{i_{n-1}})) \le \frac{\nu(R^n)}{e^{-Pn+S_n\phi(x)}} \le K_2\nu(f(R_{i_{n-1}}))$$

for any $x \in R^n$.

We observe that $\nu(f(R_i)) > 0$ for every $i = 1,\dots,m$. Indeed, consider any i fixed. Since f is transitive, there exists k_i such that $\Lambda \subset R_1 \cup \ldots \cup R_m \subset f^{k_i}(R_i)$ and, consequently, $f^{k_i}(R_i)$ has total ν-measure. So, using the fact that ν has a Jacobian, it follows that $\nu(f(R_i)) > 0$ is also positive, as claimed. To finish the proof, just take $K_5 = K_2(\inf \nu(f(R_i)))^{-1}$. □

Lemma 3.17 *If ϕ be any Hölder continuous potential for f, then*

$$\log r(\mathcal{L}_\phi) \ge P_H(\phi). \tag{14}$$

Proof If $P = \log r(\mathcal{L}_\phi)$, by Proposition 3.16 we have that for every n and $x \in R^n \in \mathcal{R}_h^n$,

$$K_5^{-1}\nu(R^n) \le \exp(S_n\phi(x) - Pn) \le K_5\nu(R^n).$$

In view of Lemma 3.7, we just need to prove that $P \ge P_H(\phi,\mathcal{U}^k)$, where $\mathcal{U}^k$ is the cover of H by hyperbolic cylinders with length bigger than k. In fact, given a cover $\mathcal{G}_n$ of H by hyperbolic cylinders, by Besicovich's covering lemma considering a subcover if necessary, we may suppose that the elements of $\mathcal{G}_n$ overlap at most L times, L depending only of the manifold M. In other words, we may decompose $\mathcal{G}_n = \mathcal{G}_n^1 \cup \ldots \cup \mathcal{G}_n^L$ such that $\mathcal{G}_n^i \cap \mathcal{G}_n^j = \emptyset$, for all $i \ne j$. Summing the inequalities above over all $R^n \in \mathcal{G}_n$, we get

$$\begin{aligned} 0 < K_5^{-1}\nu(H) &\le \sum_{U\in\mathcal{G}_n} \exp(S_{n(U)}\phi(U) - Pn(U)) \\ &\le K_5 \sum_{i=1}^{L} \nu\Bigg(\bigcup_{R\in\mathcal{G}_n^i} R\Bigg) \le K_5 L\nu(\Lambda) < +\infty. \end{aligned} \tag{15}$$

Thus, we have that the pressure $P_H(\phi,\mathcal{U}^k)$ is less or equal to P. Taking the limit when k goes to infinity, we finish the proof. □

Definition 3.18 We say that a probability ν is an *f-expanding* measure if $\nu(H) = 1$.

Lemma 3.19 *If the probability ν is a conformal measure associated to the potential ϕ that satisfy (A_2), then ν is an f-expanding measure.*

Proof Observe that by Borel-Cantelli's Lemma, we just need to prove that

$$\sum_{n=0}^{\infty} \sum_{C \in \mathcal{Q}_n(c)} \nu(C) < \infty. \tag{16}$$

It follows from Lemma 3.14 that $J_\nu f^n(x) = e^{Pn - S_n\phi(x)}$. For every n-cylinder $C = R(i_0, \dots, i_{n-1})$ we have

$$1 \geq \nu(f^n(C)) = \int_C e^{Pn - S_n\phi(x)}\, d\nu \geq e^{Pn} \inf_{x \in C} e^{-S_n(x)} \nu(C).$$

Then,

$$\nu(C) \leq e^{-Pn + S_n\phi(C)}.$$

By assumption (A_2) we have for $c_1 < P(\phi) - d \log \omega$ that

$$\sum_{C \in \mathcal{Q}_n(c)} \nu(C) \leq e^{(-P + c_1)n}.$$

By Lemma 3.17, we have that $P \geq P_H(\phi)$. Since $P_H(\phi) = P(\phi)$ by Corollary 3.11 and $c_1 < P(\phi)$ by hypothesis (A_2), we have that

$$\sum_{n=0}^{\infty} \sum_{C \in \mathcal{Q}_n(c)} \nu(C) < \sum_{n=0}^{\infty} e^{(-P(\phi) + c_1)n} < \infty.$$

The proof of lemma is complete. □

Corollary 3.20 *The pressure $P(\phi)$ is equal to* $\log r(\mathcal{L}_\phi)$.

Proof We just need to recall that if $P = \log r(\mathcal{L}_\phi)$, using that $\nu(H) > 0$ in (15) we get that $P = P_H(\phi, \mathcal{U}_k)$. Taking the limit when k goes to infinity and observing that by Corollary 3.11 we have that $P_H(\phi) = P(\phi)$, we finish the proof. □

Corollary 3.21 *The support of ν coincides with the closure of H.*

Proof Observe that by Lemma 3.19 we have that $\nu(H) = 1$ and this imply that the support of ν is contained in $\overline{H}$. Conversely, take any $x \in H$ and observe that by the definition of H, there exists a sequence $n_i = n_i(x) \in \mathbb{N}$ such that the cylinders $R^{n_i}(x)$ are hyperbolic. By Proposition 3.16, $\nu(R^{n_i}(x)) \geq \exp(S_n\phi(x) - Pn)K_5^{-1} > 0$. Since the diameters of $R^{n_i}(x)$ converge to zero, we have that any neighborhood of x has positive ν measure and, thus, $\overline{H}$ is contained in support of ν. □

4 Proof of Theorem 2.1

To construct an equilibrium measure absolutely continuous with respect to ν, we define an *map* F from f by

$$F : H \to H, \qquad F(x) = f^{n_1(x)}(x), \tag{17}$$

where $n_1(x)$ is defined as the smallest number k such that $x \in R^k$ and R^k is a hyperbolic cylinder. In this section, $n_k(x)$ denotes the k-th hyperbolic time for x.

Note that F sends any of such cylinders (intersected with H) injectively onto its image. More generally, for each $k \geq 1$ and $x \in H$ there is a largest cylinder containing x such that F^k is injective on that cylinder. We denote by F_x^{-k} the corresponding inverse branch of F^k.

Lemma 4.1 *For every y, z in the domain of F_x^{-1}*

$$d(F_x^{-1}(y), F_x^{-1}(z)) \leq e^{-cn_1(x)} d(y, z)$$

Moreover, there exists $K_6 > 0$ depending only on f such for every $k \geq 1$, every inverse branch F_x^{-k}, and every measurable subsets A, B of the domain of F_x^{-k} with $\nu(B) > 0$,

$$K_6^{-1} \frac{\nu(A)}{\nu(B)} \leq \frac{\nu(F_x^{-k}(A))}{\nu(F_x^{-k}(B))} \leq K_6 \frac{\nu(A)}{\nu(B)} \tag{18}$$

Proof The first item is a direct consequence of Lemma 3.4. For the second item, we observe that by the definition of Jacobian:

$$\frac{\nu(A)}{\nu(B)} = \frac{\int_{F_x^{-k}(A)} e^{S_{n_k(x)}\phi(x)} d\nu}{\int_{F_x^{-k}(B)} e^{S_{n_k(x)}\phi(x)} d\nu}.$$

On the other hand, by Corollary 3.5 we have that there exists $K_2 > 0$ such that $K_2^{-1} \leq \frac{e^{S_{n_k}\phi(x)}}{e^{S_{n_k}\phi(y)}} \leq K_2$, for all x, y in the domain of F_x^{-k}. Replacing in the expression above, we finish the proof. □

We point out that F is not necessarily surjective. Denote by H_F the set

$$H_F = \bigcap_{i=0}^{\infty} F^i(H).$$

Since $f^{n_1(x)}$ sends $R^{n_1(x)}(x)$ onto a rectangle, we have that

$$H_F = \left(R_{a_1} \cup \ldots \cup R_{a_s}\right) \cap H,$$

for some $a_j \in \{1, \ldots, m\}$. Indeed, if $x \in H_F$, there exists a sequence of hyperbolic cylinders R^{n_i} such that $x \in f^{n_i}(R^{n_i})$. Observe that if R^n is a n-cylinder such that $f^n(R^n) \cap \operatorname{int} R_i \neq \emptyset$, we have that $R_i = f^n(R^n)$. Thus, $x \in R_{a_j}$ for some rectangle R_{a_j} such that $R_{a_j} \cap H \subset H_F$.

We claim that for every pair R_{a_j} and R_k, we may find a n-hyperbolic cylinder $R(k, i_1, \ldots, i_{n-2}, a_j) \subset R_k$. In fact, by transitivity, we may consider n_0 bigger enough in such way that $f^{n_0}(R_k)$ contains $R_1 \cup \ldots \cup R_m$. Since $R_{a_j} \cap H_F$ is non-empty, we may find n as big as we want and a $(n - n_0)$-hyperbolic cylinder $R(i_{n_0}, \ldots, i_{n-2}, a_j)$. Since the derivative of $\|Df^{-n_0}\|$ is bounded from above, taking n big enough, and choosing $R(k, i_1, \ldots, i_{n_0})$ a non-empty cylinder in R_k, we have that $R(k, i_1, \ldots, i_{n_0}, \ldots, i_{n-2}, a_j)$ is a hyperbolic time. Therefore, given R_{a_i} and R_{a_j}, we may define:

$$l_{ij} = \min\{k \in \mathbb{N};\ \text{such that } F^k(R_{a_i} \cap H_F) \cap R_{a_j} \neq \emptyset\}.$$

Consider

$$l = \sup\{l_{ij}; 1 \le i, j \le s\}.$$

In particular, for every i, j there exists $k \le l$ such that $F^k(R_{a_i}) \cap R_{a_j} \neq \emptyset$. Observe that $\nu(R_j) > 0$ for all $j \in \{1, 2, \dots, m\}$. Define ν_F by

$$\nu_F(A) = \nu(A \cap H_F).$$

We recall that the *push-forward* of ν with respect to F^n is the measure defined by $F^n_\star \nu(A) = \nu(F^{-n}(A))$.

Lemma 4.2 *There exists a constant $K_7 > 0$ such that for $i, j \in \{1, 2, \dots, s\}$ and $k \in \mathbb{N}$,*

$$F^k_\star \nu(R_{a_i}) \le K_7 F^{k+l_{ij}}_\star \nu(R_{a_j}).$$

Proof Given $x \in F^{-k}(R_{a_i})$ and $A \subset R_{a_i}$, we consider the inverse branch F^{-k}_x. By Lemma 4.1 above

$$K_6^{-1} \frac{\nu(A)}{\nu(R_{a_i})} \le \frac{\nu\big(F^{-k}_x(A)\big)}{\nu\big(F^{-k}_x(R_{a_i})\big)} \le K_6 \frac{\nu(A)}{\nu(R_{a_i})} \tag{19}$$

and adding over all inverse branches

$$K_6^{-1} \frac{\nu(A)}{\nu(R_{a_i})} \le \frac{F^k_\star \nu(A)}{F^k_\star \nu(R_{a_i})} \le K_6 \frac{\nu(A)}{\nu(R_{a_i})}.$$

By definition, $F^{l_{ij}}$ send a l_{ij}-cylinder $R_{ij} \subset R_{a_i}$ onto R_{a_j}. In particular,

$$F^k_\star \nu(R_{ij}) \le F^{k+l_{ij}}_\star \nu(R_{a_j}).$$

Taking $A = R_{ij}$ in the (19) above, we obtain that

$$F^k_\star \nu(R_{a_i}) \le K_7 F^{k+l_{ij}}_\star \nu(R_{a_j}),$$

for $K_7 = \sup_{i,j} K_6 \nu(R_{a_j})/\nu(R_{ij})$ and this finish the proof. $\square$

Corollary 4.3 *There exists $K_8 > 0$ such that $\frac{1}{n}\sum_{k=0}^{n-1} F^k_\star \nu(R_{a_j}) > K_8 > 0$ for $n \in \mathbb{N}$ and $j \in \{1, 2, \dots, s\}$.*

Proof By the lemma above, given $j \in \{1, 2, \dots, s\}$

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} F^k_\star \nu(R_{a_i}) &\le K_7 \frac{1}{n}\sum_{k=0}^{n-1} F^{k+l_{ij}}_\star \nu(R_{a_j}) \\ &= K_7 \frac{1}{n}\sum_{k=0}^{n+l_{ij}-1} F^k_\star \nu(R_{a_j}) - K_7 \frac{1}{n}\sum_{k=0}^{l_{ij}-1} F^k_\star \nu(R_{a_j}). \end{aligned}$$

Summing for $i \in \{1, 2, \dots, s\}$ and taking n big enough,

$$\sum_{i=1}^{s} \frac{1}{n}\sum_{k=0}^{n-1} F^k_\star \nu(R_{a_i}) \le s K_5 \frac{1}{n}\sum_{k=0}^{n+l-1} F^k_\star \nu(R_{a_j}) - s K_5 \frac{1}{n}\sum_{k=0}^{l_{ij}-1} F^k_\star \nu(R_{a_j}).$$

Since the last term in the left side of the expression above goes to zero when n goes to infinity and

$$\sum_{i=1}^{s}\frac{1}{n}\sum_{k=0}^{n-1}F_\star^k\nu(R_{a_i})=\nu_F(H_F)=1,$$

we finish the proof. □

Corollary 4.4 *Every accumulation point μ_F of the sequence*

$$\mu_n=\frac{1}{n}\sum_{k=0}^{n-1}F_\star^k\nu_F$$

is an F-invariant measure absolutely continuous with respect to ν_F, with density $h=d\mu_F/d\nu_F$ bounded from zero and from infinity.

Proof It follows from (18) that

$$K_9^{-1}\nu(A)\le\frac{1}{n}\sum_{k=0}^{n-1}F_\star^k\nu(A)\le K_9\nu(A),$$

for some constant K_9 and every measurable set $A\subset R_{a_j}$. Indeed, observe that $\nu(F^{-k}(A))$ is the sum of the terms $\nu(G(A))$ over all inverse branches $G=G_k:R_{a_j}\to R(i_0,\dots,i_{n_k-2},a_j)$ of F^k and the same for $B=R_{a_j}$. As in the proof of Lemma 4.1, considering $K_7=\inf_{i=1,\dots,m}\nu(R_i)>0$, summing over all inverse branches and observing the (18):

$$K_2^{-1}\nu(A)\le\frac{\sum_{k=0}^{n-1}F_\star^k\nu(A)}{\sum_{k=0}^{n-1}F_\star^k\nu(B)}\le K_2K_7^{-1}\nu(A),$$

which implies that $K_2^{-1}K_8\nu(A)\le\frac{1}{n}\sum_{k=0}^{n-1}\nu(F^{-k}(A))\le K_2K_7^{-1}\nu(A)$. We may take for $K_9=\max\{K_1K_7^{-1},K_2K_8^{-1}\}$. From this, follows immediately that for every measurable set $A\subset H_F$

$$K_9^{-1}\nu(A)\le\mu_n(A)\le K_9\nu(A) \tag{20}$$

and from a well-known fact from measure theory it follows that any accumulation point μ_F of μ_n satisfy same inequalities. These implies that μ_F is absolutely continuous with respect to ν and its density $h=\frac{d\mu_F}{d\nu_F}$ is bounded from below and above in H by uniform constants. □

Define H_n as the set of points whose first hyperbolic time is equal to n and $H_0=R_1\cup\ldots\cup R_m$. If we define a measure μ_ϕ by

$$\mu_\phi(A)=\sum_{n=0}^{\infty}\sum_{i>n}\mu_F(f^{-n}(A)\cap H_i), \tag{21}$$

for every measurable set $A\subset R_1\cup R_2\cup\ldots\cup R_m$. Now, we prove that

Lemma 4.5 *The measure μ_ϕ is finite, f-invariant, absolutely continuous with respect to ν, with density bounded from zero and infinity. In particular, μ is expanding with integrable* *first hyperbolic time and there exists* $K_{10} > 0$ *such that for every* n *and every* $x \in R^n \in \mathcal{R}_h^n$,

$$K_{10}^{-1} \leq \frac{\mu_\phi(R^n)}{\exp(S_n\phi(x) - P(\phi)n)} \leq K_{10}. \tag{22}$$

Proof **Invariance:** If A is any Borel set, by the definition of μ_ϕ

$$\mu_\phi(f^{-1}(A)) = \sum_{n=0}^{\infty}\sum_{i>n}\mu_F(f^{-(n+1)}(A)\cap H_i) = \sum_{n=1}^{\infty}\sum_{i\geq n}\mu_F(f^{-n}(A)\cap H_i).$$

We may write the above expression as

$$\mu_\phi(f^{-1}(A)) = \sum_{n=1}^{\infty}\sum_{i>n}\mu_F(f^{-n}(A)\cap H_i) + \sum_{n=1}^{\infty}\mu_F(f^{-n}(A)\cap H_n).$$

Observe that since $\{H_i\}$ is a partition mod 0 of H:

$$\mu_F(A) = \sum_{i\geq 1}\mu_F(A\cap H_i)$$

and using that μ_F is F-invariant, we have that

$$\mu_F(A) = \mu_F(F^{-1}(A)) = \sum_{n=1}^{\infty}\mu_F(f^{-n}(A)\cap H_n).$$

Therefore,

$$\mu_\phi(f^{-1}(A)) = \sum_{n=1}^{\infty}\sum_{i>n}\mu_F(f^{-n}(A)\cap H_i) + \sum_{i=1}^{\infty}\mu_F(A\cap H_i) = \mu_\phi(A).$$

and this proves that μ_ϕ is f-invariant.

Absolute continuity and bounds for the density: In order to bound the density of μ_ϕ with respect to ν from below, we observe that it follows from the definition of μ_ϕ that for any $A \subset H_F$, by Corollary 4.4 we have

$$K_9^{-1}\nu(A) \leq \mu_F(A) \leq \mu_\phi(A) = \sum_{n=0}^{\infty}\mu_F(f^{-n}(A)\cap L_n),$$

where $L_n = \bigcup_{i>n} H_i$.

In general, if $A \subset R_i$ and $R_i \cap H_F = \emptyset$, by the transitivity of f we may choose n_0 big enough such that given any rectangle R_{a_j}, there exists a n_0-cylinder $R^{n_0} = R(i_0, \ldots, i_{n_0-1}) \subset R_{a_j}$ such that $f^{n_0-1}(R^{n_0}) = R_i$. Since $R_i \cap H_F = \emptyset$, we have that R^{n_0} is not a hyperbolic cylinder. Take $k_0 = \max\{k < n_0;\, R(i_0, \ldots, i_{k-1}) \text{ is a hyperbolic cylinder}\}$. Observe that $k_0 < n_0$ and $R^{n_0-k_0} = R(i_{k_0}, \ldots, i_{n_0-1}) \cap H$ is contained in H_F, since $F^{n_0-k_0}(H_F) \subset H_F$. On the other hand, by the definition of k_0, we have that $R^{n_0-k_0} \subset L_{n_0-k_0}$. As a consequence, we define $B = f^{-(n_0-k_0)}(A) \cap R^{n_0-k_0}$ and observe that

$$\mu_\phi(A) \geq \mu_F(f^{-(n_0-k_0)}(A)\cap L_{n_0-k_0}) \geq \mu_F(B) \geq K_9^{-1}\nu(B)$$

Since $\nu(A)/\nu(B)$ is bounded from above by a constant that depends only on n_0, ϕ and f, we get the bound from below. To obtain the bound from above, we observe that by definition of μ_ϕ

$$\mu_\phi(A) = \sum_{n=0}^{\infty} \mu_F(f^{-n}(A) \cap L_n) \leq K_9 \sum_{n=0}^{\infty} \nu(f^{-n}(A) \cap L_n),$$

where we used in the last inequality that μ_F has density with respect to ν bounded from above. To finish, we observe that $L_n \subset \bigcup_{C \in \mathcal{Q}_n(c)} C$. If $C \in \mathcal{Q}_n(c)$, using the Jacobian Formula

$$\nu(A) \geq \nu(f^n(f^{-n}(A) \cap C)) \geq \nu(f^{-n}(A) \cap C) e^{P(\phi)n - S_n(C)},$$

or

$$\nu(f^{-n}(A) \cap C) \leq e^{-P(\phi)n + S_n(C)} \nu(A).$$

As a consequence

$$\nu(f^{-n}(A) \cap L_n) \leq \nu(A) \sum_{C \in \mathcal{Q}_n(c)} e^{-P(\phi)n + S_n(C)} \leq e^{-d \log \omega n} \nu(A),$$

by Hypothesis (A_2). Adding over n, we get that $\mu_\phi(A) \leq K \nu(A)$ for $K = \sum_{n=1}^{\infty} e^{-dn \log \omega} < \infty$.

Finiteness: By condition (A_2) we have that

$$\int n_1 d\nu = \sum_{n=0}^{\infty} \nu(L_n) \leq \sum_{n=0}^{\infty} \sum_{C \in \mathcal{Q}_n(c)} \nu(C) \leq \sum_{C \in \mathcal{Q}_n(c)} e^{-P(\phi)n + S_n(C)} < \infty.$$

Since the density of μ_ϕ with respect to ν is bounded from above and $\mu_\phi(M) = \int n_1(x) d\mu_\phi(x)$, we have that $\mu_\phi(\Lambda) < +\infty$. □

Lemma 4.6 *If η is an invariant expanding measure with integrable first hyperbolic time then the sequence of hyperbolic times is non-lacunary at η-almost every point.*

Proof See [13, Proposition 3.8]. □

Corollary 4.7 *For μ_ϕ almost every point in M, there exists a sequence $K_n(x)$ such that for all $n \geq 1$ we have that*

$$K_n^{-1}(x) \leq \frac{\mu_\phi(R^n(x))}{e^{S_n(x) - nP(\phi)}} \leq K_n(x), \tag{23}$$

and $\lim_{n\to\infty} \frac{1}{n} \log K_n(x) = 0$.

Proof By Lemma 4.6 above, for μ_ϕ almost every $x \in H$, the sequence $n_1(x) < n_2(x) < \ldots$ of hyperbolic times of x is non-lacunary. We set $n_0(x) = 1$ for all x and given $n \geq 1$, if $n_k(x) \leq n \leq n_{k+1}(x)$ we define

$$M_n(x) = \max\left\{ \sup_{1 \leq l \leq n_{k+1} - n_k} e^{-l \max \phi + P(\phi) l}, K_5 \right\},$$

where K_5 is the constant obtained at Proposition 3.16. It is clear that $M_n(x)$ has subexponential growth:

$$\lim_{n\to\infty}\frac{1}{n}\log M_n(x)\le\lim_{k\to\infty}\frac{1}{n_k(x)}\big(n_{k+1}(x)-n_k(x)\big)\big(P(\phi)-\max\phi\big)=0.$$

Observe that since $n\ge n_k$, we have that $\mu_\phi(R^n(x))\le\mu_\phi(R^{n_k}(x))$. Using that $S_n\phi(x)=S_{n_k}\phi(x)+S_{n-n_k}\phi(f^{n_k}(x))$, we obtain

$$\frac{\mu_\phi(R^{n_k}(x))e^{-S_{n-n_k}\phi(f^{n_k}(x))+(n-n_k)P(\phi)}}{e^{S_{n_k}(x)-n_kP(\phi)}}\ge\frac{\mu(R^n(x))}{e^{S_n(x)-nP(\phi)}}.$$

Since $M_n(x)\ge e^{-S_{n-n_k}\phi(f^{n_k}(x))+(n-n_k)P(\phi)}$, by Proposition 3.16 we have that

$$K_5M_n(x)\ge\frac{\mu(R^n(x))}{e^{S_n(x)-nP(\phi)}}.$$

In a complete similar approach using $R^{n_{k+1}}(x)$, we get that

$$\frac{\mu(R^n(x))}{e^{S_n(x)-nP(\phi)}}\ge(K_5M_n(x))^{-1}.$$

We just define $K_n(x)=K_5M_n(x)$ to complete the proof. □

Corollary 4.8
If μ_ϕ is the non-lacunary Gibbs measure constructed in Lemma 4.5, then

$$h_{\mu_\phi}(f)+\int\phi\,d\mu_\phi=P(\phi).$$

Proof Using the Ergodic Decomposition Theorem (see [17, p. 153]), we may assume that μ_ϕ is ergodic. By Shannon-McMillan-Breiman Theorem (see [17, p. 93]) and using (23) above, for μ_ϕ almost every x:

$$\begin{aligned}h_{\mu_\phi}(f,\mathcal{R})&=\lim_{n\to\infty}-\frac{1}{n}\log\mu_\phi(R^n(x))\\&\ge\lim_{n\to\infty}-\frac{1}{n}\log K_n(x)+\lim_{n\to\infty}-\frac{1}{n}S_n\phi(x)+P(\phi).\end{aligned}$$

Taking a generic point x, observing that by Birkhoff Ergodic Theorem:

$$h_{\mu_\phi}(f)\ge h_{\mu_\phi}(f,\mathcal{R})\ge-\int\phi\,d\mu_\phi+P(\phi).$$

The opposite inequality follows from the variational principle. This finish the proof of the corollary. □

The proof of Theorem 2.1 is complete.

5 Examples

In this section, we produce some examples that satisfy conditions of Theorem 2.1. For sake of simplicity, we begin describing a one-dimensional example. We do not use the one-dimension character here and this example can be easily extended to higher dimensions.

Example 5.1 (Non-uniformly expanding repellers) Let $R_1, R_2, \ldots, R_m$ be a collection of closed intervals in [0, 1] two-by-two disjoint. Take $f_i : R_i \to [0, 1]$ any C^1 diffeomorphism. We set $\delta_i = \min_{x \in R_i} |f'(x)|$ and assume that

- There exists $p < m$ such that $\delta_i > 1$ for $i = 1, 2, \ldots, p$ (expanding regions);
- $\delta_i > 0$ for all $i \in \{1, 2, \ldots, m\}$ (no critical points).

Consider $f : R_1 \cup \ldots \cup R_m \to [0, 1]$ defined by $f(x) = f_i(x)$, if $x \in R_i$. We restrict f to the limit set Λ defined by

$$\Lambda = \bigcap_{i=0}^{\infty} f^{-i}(R_1 \cup \ldots \cup R_m).$$

It is easy to check that for all cylinders $R(i_0, \ldots, i_{n-1})$,

$$\psi_n(R(i_0, \ldots, i_{n-1})) \leq \sum_{j=0}^{n-1} \log \delta_{i_j}.$$

In particular, it is possible to estimate from above the number of cylinders in $\mathcal{Q}_n(c)$. Indeed, we may find constants γ and $c > 0$, depending only on $(\delta_1, \delta_2, \ldots, \delta_m)$. Note if we fix δ_i for $i = 1, 2, \ldots, p$ and take δ_i close to 1 for $i = p + 1, \ldots, m$, the constant γ could be chosen close to zero.

$$\#\{0 \leq i \leq n - 1;\ f^i(x) \in R_1 \cup \ldots \cup R_p\} > \gamma n,$$

then $\psi_n(R^n(x)) > c$. In particular, the rate of exponential increase of the number of cylinders in $\mathcal{Q}_n(c)$ is bounded from above by some constant $P(\gamma) < \log m$ (see [13], Lemma 3.1 for a detailed proof). Taken δ_i close to one for $i = p + 1, \ldots, m$, we have that $P(\gamma) < \log m - \log \omega$.

Observe that the topological entropy of f is greater or equal to $\log m$. In particular, the observation above mean that

$$\#\{C \in \mathcal{Q}_n(c)\} \leq e^{P(\gamma)n}$$

and $P(\gamma) < P(0) = h_{top(f)}$. Thus, the potential $\phi = 0$ satisfy (A_2). It is immediate to check using the continuity of $P(\phi)$ and the equation above, that any potential close enough to zero satisfy (A_2) as well.

Example 5.2 (Hopf-like bifurcations) This example is more involved and it will appear with detailed proofs in another work (see [7]). We describe the potentials and transformations satisfying (A_1) and (A_2) that arise from applied sciences. Hopf bifurcations occur in the Hodgkin-Huxley model for nerve membrane, the Selkov model of glycolysis, the Belousov-Zhabotinsky reaction, among other natural phenomena. Let us describe this family of maps:

Let $G : M \to M$ be a linear endomorphism on the 2-dimensional torus $M = \mathbb{T}^2$ with two complex conjugate eigenvalues $\sigma e^{\pm i\alpha}$ with $\sigma > 3$. We assume the non-resonance condition $k\alpha \notin 2\pi\mathbb{Z}$ for $k = 1, 2, 3, 4$. In cylindrical coordinates (ρ, θ) defined close to $(0, 0)$, we have

$$G(\rho, \theta) = (\sigma\rho, \theta + \alpha).$$

Similarly to [9, Sect. 1.1] we derive from G a family of endomorphisms $(\hat{f}_\mu)_{\mu\in[-1,1]}$ going through a Hopf bifurcation at $\mu = 0$. We consider a C^∞ real valued function $\Phi(\mu, w)$ defined on $[-1, 1]^2$ such that, for some $C_0 > 0$ and some small $\delta_0 > 0$,

(C_1) $\Phi(\mu, 0) = 1 - \mu \le \Phi(\mu, w)$ for all $w \ge 0$.
(C_2) $\Phi(\mu, w) = \sigma$ when $w \ge \delta_0$.
(C_3) $0 < \partial_w\Phi(\mu, w) \le C_0/\delta_0$ when $0 \le w < \delta_0$.
(C_4) There exist $\sigma_1 \in (1, \sigma)$ and $\delta_1 \in (0, \delta_0)$ such that $\Phi(\mu, w) > \sigma_1$ for all $w \ge \delta_1$ and $\partial_w\Phi(\mu, w) \ge \partial_w\Phi(\mu, 0)$ for all $w \in [0, \delta_1]$.

Since G is an expanding endomorphism, there exist a Markov partition $\mathcal{R} = \{R_1, \ldots, R_m\}$ of M for G. Let us consider that the origin is contained in the interior of some rectangle of $\mathcal{R}$. We take $\delta_0 > 0$ to be small enough so that the domain $\{(\rho, \theta) : \rho^2 \le \delta_0\}$ is contained in a small open neighborhood $V \subset M$ of the origin, such that the closure of V is itself contained in the interior of $\mathcal{R}$.

We obtain an 1-parameter family $\hat{f}_\mu$ of endomorphisms coinciding with G outside V by deforming G inside V in such a way that the restriction to V in cylindrical coordinates is given by

$$\hat{f}_\mu(\rho, \theta) = (\Phi(\mu, \rho^2)\rho, \theta + \alpha). \tag{24}$$

Note that the origin is a fixed point of $\hat{f}_\mu$ for all μ. In [9] the authors prove that this fixed point goes through a generic Hopf bifurcation at $\mu = 0$: for $\mu > 0$ the fixed point becomes an attractor. Moreover, it follows that any family $(f_\mu)_\mu$ C^5-close to $(\hat{f}_\mu)_\mu$ has a unique curve of fixed points $(p_\mu)_\mu$ close to the origin, and these fixed points also go through a Hopf bifurcation at some parameter μ_* (depending continuously of the family) close to zero: p_μ change from repelling to attracting fixed point when μ goes through μ_*.

The complement Λ_μ of the basin of attraction of the attracting point p_μ is a repeller. It follows from results of Horita and Viana in [8] that the limit capacity (and then the Hausdorff dimension) of Λ_μ is strictly less than 2. In [7], we prove that if $(f_\mu)_\mu$ is a family of endomorphism in a C^5-neighborhood of $(\hat{f}_\mu)_\mu$. Then, $(f_\mu)_\mu$ satisfy condition (A_1) and for μ close to μ_*, we may find an open interval $(\gamma_\mu^0, \gamma_\mu^1)$ such that for every $\gamma \in (\gamma_\mu^0, \gamma_\mu^1)$, the function $\phi(x) = -\gamma \log|\det Df(x)|$ satisfy (A_2) and, as a consequence, has an non-lacunary Gibbs measure.

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